Th 1 (MCT) Let $(x_n)\uparrow$, increasing seq of real nos. Then

(i) If (x_n) is bounded (so $\alpha := \sup\{x_n : n \in \mathbb{N}\}$ exists in $\mathbb{R}$) then $\lim x_n = \alpha$

(ii) If (x_n) is not bounded then $\lim_n x_n = +\infty$, i.e. $\forall M > 0 \; \exists N \in \mathbb{N}$ s.t.

$$x_n \geq M \quad \forall n \geq N. \qquad (\#)$$

proof. Let $\varepsilon > 0$. Then $\alpha - \varepsilon < x_N$ for some $N \in \mathbb{N}$. Since $(x_n)\uparrow$ and α is an u.b. of $\{x_n : n \in \mathbb{N}\}$ it follows that $\forall n \geq N$

$$\alpha - \varepsilon < x_N \leq x_n \leq \alpha$$

so $-\varepsilon < x_n - \alpha \leq 0$, and $|x_n - \alpha| < \varepsilon \; \forall n \geq N$.

(i) is proved.

(ii). Let $M \in \mathbb{R}$. Then, as (x_n) is not bounded, M is not an upper bd of the seq, and so $\exists N \in \mathbb{N}$ s.t. $x_N \geq M$.

Thus $(\#)$ holds as $(x_n)\uparrow$.

Th 2 (MCT for decreasing seq).

...... (pl. fill in details).

Example 1. Let $x_1 = 1$ and $x_{n+1} := \frac{1}{3}(2x_n + 5)$ $\forall n$.

Then, by MI, $(x_n)\uparrow$ and bounded above by 100:

$x_{k+1} \geq \cancel{x}_k \Rightarrow 2x_{k+1} \geq 2x_k \Rightarrow \frac{2x_{k+1}+5}{3} \geq \frac{2x_k+5}{3}$

$\Rightarrow x_{k+2} \geq x_{k+1}$, showing the increasing proped as $x_2 > x_1$.

Similarly you can prove that $x_n \leq 100$ $\forall n$. By MCT let $x := \lim x_n$ $(\in \mathbb{R})$ (so $\lim_n x_{n+1} = x$). It follows from computation rules that $x = \frac{1}{3}(2x+5)$ i.e. $x = 5$.